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Lattices generated by orbits of subspaces under finite singular pseudo-symplectic groups I

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Let $\mathbb{F}_q^{(2\nu+1+l)}$ be the $(2\nu+1+l)$-dimensional vector space over the finite field $\mathbb{F}_q$. In the paper we assume that $\mathbb{F}_q$ is a finite field of characteristic 2, and $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$ the singular pseudo-symplectic groups of degree $2\nu+1+l$ over $\mathbb{F}_q$. Let $\mathcal{M}$ be any orbit of subspaces under $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$. Denote by $\mathcal{L}$ the set of subspaces which are intersections of subspaces in $\mathcal{M}$ and the intersection of the empty set of subspaces of $\mathbb{F}_q^{(2\nu+1+l)}$ is assumed to be $\mathbb{F}_q^{(2\nu+1+l)}$. By ordering $\mathcal{L}$ by ordinary or reverse inclusion, two lattices are obtained. This paper studies the inclusion relations between different lattices, a characterization of subspaces contained in a given lattice $\mathcal{L}$, and the characteristic polynomial of $\mathcal{L}$.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a two power, $\mathbb{F}_q^{(2\nu+1+l)}$ be the $(2\nu+1+l)$-dimensional row vector space over the finite field $\mathbb{F}_q$, and $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$ be one of the singular pseudo-symplectic groups of degree $2\nu+1+l$ over $\mathbb{F}_q$. There is an action of $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu+1+l)}$ defined as follows:

$$\begin{aligned} &\mathbb{F}_q^{(2\nu+1+l)} \times Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q) \to \mathbb{F}_q^{(2\nu+1+l)},\\ &((x_1,\ldots,x_{2\nu},x_{2\nu+1},\ldots,x_{2\nu+1+l}),T) \mapsto (x_1,\ldots,x_{2\nu},x_{2\nu+1},\ldots,x_{2\nu+1+l})T. \end{aligned} \tag{1}$$

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Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+1+l)}$ $(1 \leqslant m \leqslant 2\nu+1+l)$, and $v_1, v_2, \ldots, v_m$ be a basis of P. Then the $m \times (2\nu+1+l)$ matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

is called a matrix representation of P. We usually denote a matrix representation of the m-dimensional subspace P still by P. The above action induces an action on the set of subspaces of $\mathbb{F}_q^{(2\nu+1+l)}$, i.e., a subspace P is carried by $T \in Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$ into the subspace PT. The set of subspaces of $\mathbb{F}_q^{(2\nu+1+l)}$ is partitioned into orbits under $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$. Clearly, $\{0\}$ and $\{\mathbb{F}_q^{(2\nu+1+l)}\}$ are two trivial orbits. Let $\mathcal{M}$ be any orbit of subspaces under $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$. Denote the set of subspaces which are intersections of subspaces in $\mathcal{M}$ by $\mathcal{L}(\mathcal{M})$ and call $\mathcal{L}(\mathcal{M})$ the set of subspaces generated by $\mathcal{M}$. We agree that the intersection of an empty set of subspaces is $\mathbb{F}_q^{(2\nu+1+l)}$. Then $\mathbb{F}_q^{(2\nu+1+l)} \in \mathcal{L}(\mathcal{M})$. Partially ordering $\mathcal{L}(\mathcal{M})$ by ordinary or reverse inclusion, we get two posets and denote them by $\mathcal{L}_O(\mathcal{M})$ or $\mathcal{L}_R(\mathcal{M})$ respectively. Clearly, for any two elements $P, Q \in \mathcal{L}_O(\mathcal{M})$,

$$P \wedge Q = P \cap Q, \quad P \vee Q = \cap\{R \in \mathcal{L}_O(\mathcal{M}) : R \supseteq \langle P, Q\rangle\},$$

where $\langle P, Q\rangle$ is the subspace generated by P and Q. Therefore, $\mathcal{L}_O(\mathcal{M})$ is a finite lattice.

Similarly, for any two elements $P, Q \in \mathcal{L}_R(\mathcal{M})$,

$$P \wedge Q = \cap\{R \in \mathcal{L}_R(\mathcal{M}) : R \supseteq \langle P, Q\rangle\}, \quad P \vee Q = P \cap Q,$$

so $\mathcal{L}_R(\mathcal{M})$ is also a finite lattice. Both $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$ are called the lattices generated by $\mathcal{M}$.

The purpose of this paper is to study the various lattices $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$ generated by different orbits $\mathcal{M}$ of subspaces under singular pseudo-symplectic group $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$. The contents of the study include the inclusion relations between different lattices, a characterization of subspaces contained in a given lattice $\mathcal{L}_R(\mathcal{M})$ (resp. $\mathcal{L}_O(\mathcal{M})$), and the characteristic polynomial of $\mathcal{L}_R(\mathcal{M})$.

The results on the lattices generated by distance-regular graphs can be found in Guo et al. [2], the lattices generated by orbits of subspaces under finite nonsingular classical groups can be found in Wang and Feng [3], Wang and Guo [4], Huo et al. [5–7], Huo and Wan [8,9], and under finite singular symplectic group and singular unitary group can be found in Gao and You [10,11]. In this paper,we study lattices generated by orbits of subspaces under finite pseudo-symplectic group.

2. Preliminaries

In the following we recall some definitions and facts on ordered sets and lattices (see [1,9]).

Let A be a partially ordered set, and $a, b \in A$. We say that b covers a and write $a < \cdot b$, if $a < b$ and there exists no $c \in A$ such that $a < c < b$. An element $m \in A$ is called a minimal element if there exists no elements $a \in A$ such that $a < m$. If A has a unique minimal element, denote it by 0 and we say that A is a poset with 0.

Let A be a poset with 0 and $a \in A$. If all maximal ascending chains starting from 0 with endpoint a have the same finite length, this common length is called the rank $r(a)$ of a. If rank $r(a)$ is defined for every $a \in A$, A is said to have the rank function $r : A \to \mathbb{N}$, where $\mathbb{N}$ is the set consisting of all positive integers and 0.

A poset A is said to satisfy the Jordan–Dedekind (JD) condition if any two maximal chains between the same pair of elements of A have the same finite length.

Proposition 2.1 (Prop. 2.1 of [1]). *Let A be a poset with* 0. *If A satisfies the JD condition then A has the rank function $r : A \to \mathbb{N}$ which satisfies*

(i) $r(0) = 0$,
(ii) $a < \cdot b \Rightarrow r(b) = r(a) + 1$.

Conversely, if A admits a function $r : A \to \mathbb{N}$ satisfying (i) *and* (ii), *then A satisfies the JD condition with r as its rank function.*

Let A be a poset with 0. An element $a \in A$ is called an atom of A if $0 < \cdot a$. A lattice L with 0 is called an atomic lattice if every element $a \in L \setminus \{0\}$ is a supremum of atoms, i.e., $a = \sup\{b \in L | 0 < \cdot b \leqslant a\}$.

Proposition 2.2. *Let L be a finite lattice with* 0. *Then L is an atomic lattice if and only if every element of* $L \setminus \{0\}$ *is a union of atoms.*

Let L be a lattice with 0. L is called a geometric lattice if

G_1 for every element $a \in L \setminus \{0\}, a = \sup\{b \in L | 0 < \cdot b \leqslant a\}$,

G_2 L possesses a rank function r and for all $x, y \in L$

$$r(x \wedge y) + r(x \vee y) \leqslant r(x) + r(y), \tag{2}$$

G_3 there does not exist infinite chains in L.

Let

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} K & \\ & 1 \end{pmatrix}, \quad S_{1,l} = \begin{pmatrix} S_1 & \\ & 0^{(l)} \end{pmatrix}.$$

The singular pseudo-symplectic group of degree $2\nu + 1 + l$ over $\mathbb{F}_q$, denoted by $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$, consists of all $(2\nu + 1 + l) \times (2\nu + 1 + l)$ nonsingular matrices T over $\mathbb{F}_q$ such that

$$TS_{1,l}T^t = S_{1,l}.$$

If $l = 0, Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q) = Ps_{2\nu+1}(\mathbb{F}_q)$ is the pseudo-symplectic group of degree $2\nu + 1$. An m-dimensional subspace P is said to be of type $(m, 2s + \tau, s, \varepsilon)$,where $\tau = 0, 1, 2$ and $\varepsilon = 0, 1$, if $PS_{1,l}P^t$ is cogredient to $\mathcal{M}(m, 2s + \tau, s)$ and P does not or does contain a vector of the form $(0, 0, \ldots, 1, x_{2\nu+2}, \ldots, x_{2\nu+1+l})$ corresponding to the cases $\varepsilon = 1$ or $\varepsilon = 0$, respectively, where the matrix representation of $\mathcal{M}(m, 2s + \tau, s)$ when $\tau = 0, 1, 2$ are as follows:

$$\mathcal{M}(m, 2s, s) = \begin{pmatrix} 0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & 0^{(m-2s)} \end{pmatrix},$$

$$\mathcal{M}(m, 2s+1, s) = \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & 1 & \\ & & & 0^{(m-2s-1)} \end{pmatrix},$$

$$\mathcal{M}(m, 2s+2, s) = \begin{pmatrix} 0 & I^{(s)} & & & \\ I^{(s)} & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 1 & \\ & & & & 0^{(m-2s-2)} \end{pmatrix}.$$

Let $e_1, e_2, \ldots, e_{2\nu+1},\ e_{2\nu+2}, \ldots, e_{2\nu+1+l}$, where

$$e_i = (0, \quad \ldots, \quad 0, \quad \underset{i}{1}, \quad 0, \quad \ldots, \quad 0)$$

be a basis of $\mathbb{F}_q^{(2\nu+1+l)}$ and denote by E the l-dimensional subspace of $\mathbb{F}_q^{(2\nu+1+l)}$ generated by $e_{2\nu+2}$, $e_{2\nu+3}, \ldots, e_{2\nu+1+l}$. An m-dimensional subspace P is called a subspace of type $(m, 2s + \tau, s, \varepsilon, k)$ if

(i) P is a subspace of type $(m, 2s + \tau, s, \varepsilon)$,
(ii) $\dim(P \cap E) = k$.

Denote the set of all subspaces of type $(m, 2s+\tau, s, \varepsilon, k)$ in $\mathbb{F}_q^{(2\nu+1+l)}$ by $\mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$. By [12, Theorem 4.16] we know that $\mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ is non-empty if and only if

$$\left.\begin{array}{l}(\tau,\varepsilon)=(0,0),(1,0),(1,1),(2,0),\\ k\leqslant l,\\ 2s+\max\{\tau,\varepsilon\}\leqslant m-k\leqslant \nu+s+[\tau/2]+\varepsilon,\end{array}\right\} \tag{3}$$

or

$$\left.\begin{array}{l}(\tau,\varepsilon)=(0,0),(1,0),(1,1),(2,0),\\ \max\{0, m-\nu-s-[\tau/2]-\varepsilon\}\leqslant k\leqslant \min\{l, m-2s-\max\{\tau,\varepsilon\}\}.\end{array}\right\} \tag{4}$$

Moreover, if $\mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ is non-empty, then it forms an orbit of subspaces under $Ps_{2\nu+1+l,2\nu+1}(\mathbb{F}_q)$. Let $\mathcal{L}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ denote the set of subspaces which are intersections of subspaces in $\mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ and agree that the intersection of an empty set of subspaces is $\mathbb{F}_q^{(2\nu+1+l)}$. Partially ordering $\mathcal{L}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ by ordinary or reverse inclusion, we get two finite lattices and denote them by $\mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ and $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ respectively.

Denote by $\mathcal{M}(m, s, k; 2\nu+l, \nu)$ the orbit formed by all m-dimensional subspaces in $\mathbb{F}_q^{(2\nu+l)}$ under $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ and denote by $\mathcal{L}(m, s, k; 2\nu+l, \nu)$ the set of subspaces which are intersection of subspaces in $\mathcal{M}(m, s, k; 2\nu+l, \nu)$. Same as above we get two finite lattices and denote them by $\mathcal{L}_O(m, s, k; 2\nu+l, \nu)$ and $\mathcal{L}_R(m, s, k; 2\nu+l, \nu)$ respectively.

3. The inclusion relations between different lattices

Lemma 3.1. *Assume that $(m, 2s+\tau, s, \varepsilon, l)$ satisfies condition (3) and (4). Then*

$$\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, l; 2\nu+1+l, 2\nu+1) \backsimeq \mathcal{L}_R(m-l, 2s+\tau, s, \varepsilon; 2\nu+1),$$

where $\backsimeq$ denotes lattice isomorphism. And the case $\mathcal{L}_R(m-l, 2s+\tau, s, \varepsilon; 2\nu+1)$ has been discussed by Wan and Huo [9]. So we only discuss the case $0\leqslant k<l$ in this paper.

Lemma 3.2. *Let $0\leqslant k<l, 2s\leqslant m-k\leqslant \nu+s$. Then, $m-k\neq 2\nu+l$ and*

$$\mathcal{L}_R(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1) \backsimeq \mathcal{L}_R(m, s, k; 2\nu+l, \nu).$$

Proof. Since $2s\leqslant m-k\leqslant \nu+s$, have $m-k\neq 2\nu+l$ and $(m, 2s, s, 0, k)$ satisfies condition (3) and (4), then $\mathcal{M}(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)\neq\emptyset$. Let $P\in\mathcal{M}(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)$ and assume that

$$P=\begin{array}{cc}\begin{pmatrix}(P_{11},0) & P_{12}\\ 0 & P_{22}\end{pmatrix} & \begin{array}{l}m-k\\ k\end{array}\\ \begin{array}{cc}\quad 2\nu+1 & \quad l\end{array} & \end{array},$$

where $(P_{11},0)\in\mathcal{M}(m-k, 2s, s, 0; 2\nu+1)$, $P_{11}\in\mathcal{M}(m-k, s; 2\nu)$.

We define a mapping

$$\psi:\mathcal{M}(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)\to\mathcal{M}(m, s, k; 2\nu+l, \nu),$$

$$P=\begin{array}{cc}\begin{pmatrix}(P_{11},0) & P_{12}\\ 0 & P_{22}\end{pmatrix} & \begin{array}{l}m-k\\ k\end{array}\\ \begin{array}{cc}\quad 2\nu+1 & \quad l\end{array} & \end{array}\mapsto Q=\begin{array}{cc}\begin{pmatrix}P_{11} & P_{12}\\ 0 & P_{22}\end{pmatrix} & \begin{array}{l}m-k\\ k\end{array}\\ \begin{array}{cc}\quad 2\nu & \quad l\end{array} & \end{array}.$$

Obviously, ψ is a bimorphism. Since $\mathcal{M}(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)$ and $\mathcal{M}(m, s, k; 2\nu+l, \nu)$ is the set of atom of $\mathcal{L}_R(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)$ and $\mathcal{L}_R(m, s, k; 2\nu+l, \nu)$ respectively, we can deduce other bimorphism

$$\varphi : \mathcal{L}_R(m, 2s, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) \to \mathcal{L}_R(m, s, k;\ 2\nu + l, \nu),$$
$$\bigcap_{i\in I} P_i \mapsto \bigcap_{i\in I} \psi(P_i)$$

and it remains the relation of partial ordering of lattice $\mathcal{L}_R(m, 2s, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1)$ and $\mathcal{L}_R(m, s, k;\ 2\nu + l, \nu)$. So φ is a lattice isomorphism. □

Lemma 3.3. *Let* $2\nu + 1 + l > m \geqslant 1, 0 \leqslant k < l, \tau = 0, 1, 2$ *and* $(m, 2s + \tau, s, 0, k)$ *satisfy*

$$2s + \tau \leqslant m - k \leqslant \nu + s + [\tau/2]. \tag{5}$$

Then

$$\mathcal{L}_R(m, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) \supset \mathcal{L}_R(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1).$$

Proof. (i) If $\tau = 0$, by Lemma 3.2 in Ref. [10] we know $\mathcal{L}_R(m, s, k;\ 2\nu + l, \nu) \supset \mathcal{L}_R(m - 1, s, k;\ 2\nu + l, \nu)$ and by Lemma 3.2 above we have the lattice isomorphism

$$\mathcal{L}_R(m, 2s, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) \backsimeq \mathcal{L}_R(m, s, k;\ 2\nu + l, \nu).$$

Thus

$$\mathcal{L}_R(m, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) \supset \mathcal{L}_R(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1).$$

(ii) $\tau > 0$. We need only to show that

$$\mathcal{L}_R(m, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) \supset \mathcal{M}(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1).$$

If $2s + \tau > m - 1 - k$, then

$$\mathcal{M}(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) = \phi,$$

thus

$$\begin{aligned}&\mathcal{L}_R(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1)\\ &\quad = \{\mathbb{F}_q^{(2\nu+1+l)}\} \subset \mathcal{L}_R(m, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1).\end{aligned}$$

Now suppose that $2s + \tau \leqslant m - 1 - k$, then

$$\mathcal{M}(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1) \neq \phi.$$

Let $P \in \mathcal{M}(m - 1, 2s + \tau, s, 0, k;\ 2\nu + 1 + l, 2\nu + 1)$ and assume that

$$P = \begin{array}{cl} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{l} (m-1) - k \\ k \end{array} \\ \begin{array}{cc} 2\nu + 1 & \quad l \end{array} & \end{array},$$

where $P_{11} S_1 P_{11}^t = M(m - 1 - k, 2s + \tau, s)$, rank $P_{22} = k$, and since $\varepsilon = 0$ we can assume that

$$P_{11} = \begin{array}{cl} \begin{pmatrix} Q_1 & 0 \\ \nu & 1 \\ Q_2 & 0 \end{pmatrix} & \begin{array}{l} 2s + [\tau/2] \\ 1 \\ m - 2 - k - 2s - [\tau/2] \end{array} \\ \begin{array}{cc} 2\nu & \quad 1 \end{array} & \end{array},$$

where rank $Q = \begin{pmatrix} Q_1 \\ \nu \\ Q_2 \end{pmatrix} = m - 1 - k$, and

$$QK_{2\nu}Q^t = \begin{cases} [K_{2s}, 0^{(m-1-k-2s)}], & \text{when } \tau = 1, \\ [K_{2s}, K_{2\cdot 1}, 0^{(m-3-k-2s)}], & \text{when } \tau = 2, \end{cases}$$

where $K_{2s} = \begin{pmatrix} 0 & I^{(s)} \\ I^{(s)} & 0 \end{pmatrix}$. So $Q \in \mathcal{M}(m - 1 - k, s + [\tau/2]; 2\nu)$, i.e. Q is a subspace of type $(m - 1 - k, s + [\tau/2])$ of $\mathbb{F}_q^{(2\nu)}$. There exist two non-zero vectors ν_1 and ν_2 such that $\begin{pmatrix} Q \\ \nu_1 \end{pmatrix}$ and $\begin{pmatrix} Q \\ \nu_2 \end{pmatrix}$ are a pair of subspaces of type $(m - k, s + [\tau/2])$ of 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, and

$$\begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ v_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ v_2 & 0 \end{pmatrix}$$

are subspaces of type $(m-k, 2s+[\tau/2], s, 0)$ of $(2\nu+1)$-dimensional pseudo-symplectic space $\mathbb{F}_q^{(2\nu+1)}$, and

$$\begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ v_1 & 0 \end{pmatrix} \cap \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ v_2 & 0 \end{pmatrix} = P_{11}.$$

Let

$$P_i = \begin{array}{c} \begin{pmatrix} P_{11} & P_{12} \\ (v_i, 0) & 0 \\ 0 & P_{22} \end{pmatrix} \end{array} \begin{array}{l} m-1-k \\ 1 \\ k \end{array} \qquad (i = 1, 2).$$

Then $P_i \in \mathcal{M}(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$ $(i=1,2), P = P_1 \cap P_2$, $\dim(P_i \cap E) = k$. So $P \in \mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$.

Lemma 3.4. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, s \geqslant 1, \tau = 0, 1, 2$ *and* $(m, 2s+\tau, s, 0, k)$ *satisfies condition* (5). *Then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m-1, 2(s-1)+\tau, s-1, 0, k; 2\nu+1+l, 2\nu+1). \end{aligned}$$

Proof. (i) If $\tau = 0$, by the Lemma 3.3 in Ref. [10] we know $\mathcal{L}_R(m, s, k; 2\nu+l, \nu) \supset \mathcal{L}_R(m-1, s-1, k; 2\nu+l, \nu)$ and by Lemma 3.2 we have the lattice isomorphism

$$\mathcal{L}_R(m-1, 2(s-1), s-1, 0, k; 2\nu+1+l, 2\nu+1) \simeq \mathcal{L}_R(m-1, s-1, k; 2\nu+l, \nu),$$

thus

$$\begin{aligned} &\mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m-1, 2(s-1)+\tau, s-1, 0, k; 2\nu+1+l, 2\nu+1). \end{aligned}$$

(ii) If $\tau > 0$, by the condition we have

$$2(s-1)+\tau \leqslant m-k-1 \leqslant \nu+s-1+[\tau/2].$$

Thus $\mathcal{M}(m-1, 2(s-1), s-1, 0, k; 2\nu+1+l, 2\nu+1) \neq \phi$.

Let $P \in \mathcal{M}(m-1, 2(s-1), s-1, 0, k; 2\nu+1+l, 2\nu+1)$, and assume that

$$P = \begin{array}{c} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} \\ \begin{array}{cc} 2\nu+1 & l \end{array} \end{array} \begin{array}{l} (m-1)-k \\ k \\ \end{array},$$

where $P_{11} \in \mathcal{M}(m-1-k, 2(s-1)+\tau, s-1, 0; 2\nu+1)$, $\operatorname{rank} P_{22} = k$, and since $\varepsilon = 0$ we can assume that

$$P_{11} = \begin{array}{c} \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \end{pmatrix} \\ \begin{array}{cc} 2\nu & 1 \end{array} \end{array} \begin{array}{l} 2(s-1)+[\tau/2] \\ 1 \\ m-k-2s-[\tau/2] \\ \end{array},$$

where $\operatorname{rank} Q = \begin{pmatrix} Q_1 \\ v \\ Q_2 \end{pmatrix} = m-1-k$, and

$$QK_{2\nu}Q^t = \begin{cases}[K_{2(s-1)}, 0^{(m-k-2s+1)}], & \text{when } \tau = 1,\\ [K_{2(s-1)}, K_{2\cdot 1}, 0^{(m-k-2s-1)}], & \text{when } \tau = 2.\end{cases}$$

So $Q \in \mathcal{M}(m-1-k, s-1+[\tau/2]; 2\nu)$, and there exist $(m-k-2(s+[\tau/2])+1) \times 2\nu$ matrix X and $2(\nu+s-m+k+[\tau/2]) \times 2\nu$ matrix Y such that

$$W = \begin{pmatrix} Q \\ X \\ Y \end{pmatrix}$$

is nonsingular matrix and

$$WK_{2\nu}W^t = \begin{cases}[K_{2(s-1)}, K_{2(m-k-2s+1)}, K_{2(\nu+s-m-k)}], & \text{when } \tau = 1,\\ [K_{2(s-1)}, K_{2\cdot 1}, K_{2(m-k-2s-1)}, K_{2(\nu+s-m-k+1)}], & \text{when } \tau = 2.\end{cases}$$

(a) If $\tau = 1$, by the condition we know $m-k-2s+1 \geqslant 2$, i.e. there are two rows in matrix X at least. Take the first row x_1 and the second row x_2 of X, then

$$\begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 + x_2 & 0 \end{pmatrix}$$

are the subspaces of type $(m-k, 2s+1, s, 0)$ of $(2\nu+1)$-dimensional pseudo-symplectic space $\mathbb{F}_q^{(2\nu+1)}$, and

$$\begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 & 0 \end{pmatrix} \cap \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 + x_2 & 0 \end{pmatrix} = P_{11}.$$

Let

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ (x_1, 0) & 0 \\ 0 & P_{22} \end{pmatrix}, P_2 = \begin{pmatrix} P_{11} & P_{12} \\ (x_1 + x_2, 0) & 0 \\ 0 & P_{22} \end{pmatrix}.$$

Then $P = P_1 \cap P_2$, since $P_1, P_2 \in \mathcal{M}(m, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1)$, thus $P \in \mathcal{M}(m, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1)$.

(b) If $\tau = 2$, then by the condition we know $m-k-2s-1 \geqslant 1$. Let x_1 be the first row of X, then

$$\begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 & 1 \end{pmatrix}$$

are the subspaces of type $(m-k, 2s+2, s, 0)$ of $(2\nu+1)$-dimensional pseudo-symplectic space $\mathbb{F}_q^{(2\nu+1)}$, and

$$\begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 & 0 \end{pmatrix} \cap \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \\ x_1 & 1 \end{pmatrix} = P_{11},$$

let

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ (x_1, 0) & 0 \\ 0 & P_{22} \end{pmatrix}, P_2 = \begin{pmatrix} P_{11} & P_{12} \\ (x_1, 1) & 0 \\ 0 & P_{22} \end{pmatrix}.$$

Then $P = P_1 \cap P_2$, since $P_1, P_2 \in \mathcal{M}(m, 2s+2, s, 0, k; 2\nu+1+l, 2\nu+1)$, thus $P \in \mathcal{M}(m, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1)$. Therefore

$$\mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{L}_R(m-1, 2(s-1)+\tau, s-1, 0, k;\ 2\nu+1+l, 2\nu+1). \qquad \square$$

By similar discussion of Lemma 3.3, 3.4 above and the Lemmas 7.9–7.11 of Ref. [9], we obtain the following lemmas:

Lemma 3.5. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, \tau = 1, 2$ *and* $(m, 2s+\tau, s, 0, k)$ *satisfies condition* (5). *Then*

$$\mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{L}_R(m-1, 2s+(\tau-1), s, 0, k;\ 2\nu+1+l, 2\nu+1).$$

Lemma 3.6. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l$ *and* $(m, 2s+2, s, 0, k)$ *satisfies condition* (5). *Then*

$$\mathcal{L}_R(m, 2s+2, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{L}_R(m-1, 2s, s, 0, k;\ 2\nu+1+l, 2\nu+1).$$

Lemma 3.7. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, s \geqslant 1$ *and* $(m, 2s+1, s, 0, k)$ *satisfies condition* (5). *Then*

$$\mathcal{L}_R(m, 2s+1, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{L}_R(m-1, 2(s-1)+2, s-1, 0, k;\ 2\nu+1+l, 2\nu+1).$$

Lemma 3.8. *Let* $2\nu+1+l > m \geqslant 1, 1 \leqslant k < l$ *and* $(m, 2s+\tau, s, \varepsilon, k)$ *satisfies*

$$2s + \max\{s, \tau\} \leqslant m-k \leqslant \nu+s+[\tau/2]+\varepsilon.$$

Then

$$\mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{L}_R(m-1, 2s+\tau, s, 0, k-1;\ 2\nu+1+l, 2\nu+1), \quad \tau = 0, 1, 2,$$

and

$$\mathcal{L}_R(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{L}_R(m-1, 2s+1, s, 1, k-1;\ 2\nu+1+l, 2\nu+1).$$

Proof. We only prove the first formula, the second can be followed in the similar way.

If $l = 1$, then $\mathcal{M}(m-1, 2s+\tau, s, 0, k-1;\ 2\nu+1+l, 2\nu+1) = \mathbb{F}_q^{(2\nu+1+l)}$. It is obvious that $\mathbb{F}_q^{(2\nu+1+l)} \subset \mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$.

Now suppose that $l \geqslant 2$, we need only to show that

$$\mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$
$$\supset \mathcal{M}(m-1, 2s+\tau, s, 0, k-1;\ 2\nu+1+l, 2\nu+1).$$

Let $P \in \mathcal{M}(m-1, 2s+\tau, s, 0, k-1;\ 2\nu+1+l, 2\nu+1)$, and assume that

$$P = \begin{array}{cc} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{l} (m-1)-(k-1) \\ k-1 \end{array} \\ \begin{array}{cc} 2\nu+1 & \quad l \end{array} & \end{array},$$

where $P_{11} \in \mathcal{M}(m-k, 2s+\tau, s, 0;\ 2\nu+1)$, $\operatorname{rank} P_{22} = k-1$.

Let $v_{2\nu+2}, v_{2\nu+3}, \ldots, v_{2\nu+k}$ be the $(k-1)$ row vectors of $(0, P_{22})$, since $\dim(P_i \cap E) = k-1$, there exist $l-(k-1) \geqslant 2$ vectors $v_{2\nu+1+k}, \ldots, v_{2\nu+1+l}$ such that

$$E = \langle v_{2\nu+2}, v_{2\nu+3}, \ldots, v_{2\nu+k}, v_{2\nu+1+k}, \ldots, v_{2\nu+1+l} \rangle.$$

Let

$$P_1 = \begin{pmatrix} P \\ v_{2\nu+k+1} \end{pmatrix}, \quad P_2 = \begin{pmatrix} P \\ v_{2\nu+k+2} \end{pmatrix},$$

then $P = P_1 \cap P_2$. Since $P_i \in \mathcal{M}(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$, thus $P \in \mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$. $\square$

Lemma 3.9. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, s \geqslant 1$ *and* $(m, 2s+1, s, 1, k)$ *satisfies* $2s+1 \leqslant m-k \leqslant \nu+s+1$. *Then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m-1, 2(s-1)+1, s-1, 1, k;\ 2\nu+1+l, 2\nu+1). \end{aligned}$$

Proof. (i) If $\nu < s$, by the condition we know $\mathcal{M}(m-1, 2(s-1)+1, s-1, 1, k;\ 2\nu+1+l, 2\nu+1) \neq \phi$.

Let $P \in \mathcal{M}(m-1, 2(s-1)+1, s-1, 1, k;\ 2\nu+1+l, 2\nu+1)$, and assume that

$$P = \begin{array}{cl} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{l} (m-1)-k \\ k \end{array} \\ \begin{array}{cc} 2\nu+1 & \quad l \end{array} & \end{array},$$

where $P_{11} S_1 P_{11}^t = M(m-1-k, 2(s-1)+1, s-1)$, rank $P_{22} = k$. Since $\varepsilon = 1$, we can assume that

$$P_{11} = \begin{array}{cl} \begin{pmatrix} Q_1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{l} m-2-k \\ 1 \end{array} \\ \begin{array}{cc} 2\nu & \quad 1 \end{array} & \end{array},$$

where rank $Q_1 = m-2-k$, and $Q_1 K_{2\nu} Q_1^t = M(m-2-k, 2(s-1), s-1)$. So there exist $(m-2-k) \times 2\nu$ matrix X and $2(\nu+s-m+k+1) \times 2\nu$ matrix Y such that

$$W = \begin{pmatrix} Q_1 \\ X \\ Y \end{pmatrix}$$

is nonsingular matrix, and

$$W K_{2\nu} W^t = [K_{2(s-1)}, K_{2(m-k-2s)}, K_{2(\nu+s-m+k+1)}].$$

If $m-k-2s \geqslant 2$, take the first row x_1 and the second row x_2 of X. Let

$$P_i = \begin{pmatrix} P_{11} & P_{12} \\ (x_i, 0) & 0 \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array} \qquad (i = 1, 2),$$

then $P_i \in \mathcal{M}(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1)$ and $P = P_1 \cap P_2 \in \mathcal{L}_R(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1)$.

If $m-k-2s = 1$, since $s < \nu$, have $\nu+s-m+k+1 = \nu-s > 0$, take y be the first row of Y,

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ (X, 0) & 0 \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array}, \quad P_2 = \begin{pmatrix} P_{11} & P_{12} \\ (X+y, 0) & 0 \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array},$$

then $P_i \in \mathcal{M}(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1)$ and $P = P_1 \cap P_2 \in \mathcal{L}_R(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1)$.

(ii) If $\nu = s$, then have $k < l, m = 2\nu + 1 + k$. Let $P \in \mathcal{M}(m-1, 2(\nu-1)+1, \nu-1, 1, k; 2\nu + 1 + l, 2\nu + 1)$, and assume that

$$P = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & H_1 & Q_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & H_2 & Q_2 \\ 0 & 0 & I^{(\nu-1)} & 0 & 0 & 0 & H_3 & Q_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & H_4 & Q_4 \\ 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0^{(l-k-1)} \end{pmatrix}.$$

Let

$$P_1 = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & H_1 & Q_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & H_2 & Q_2 \\ 0 & 0 & I^{(\nu-1)} & 0 & 0 & 0 & H_3 & Q_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & H_4 & Q_4 \\ 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0^{(l-k-1)} \end{pmatrix},$$

$$P_2 = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & H_1 & Q_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & H_2 & Q_2 \\ 0 & 0 & I^{(\nu-1)} & 0 & 0 & 0 & H_3 & Q_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & H_4 & Q_4 \\ 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0^{(l-k-1)} \end{pmatrix},$$

then $P_i \in \mathcal{M}(m, 2\nu+1, \nu, 1, k; 2\nu + 1 + l, 2\nu + 1)$ and $P = P_1 \cap P_2 \in \mathcal{L}_R(m, 2\nu + 1, \nu, 1, k; 2\nu + 1 + l, 2\nu + 1)$. $\square$

Lemma 3.10. *Let $2\nu + 1 + l > m \geqslant 1, 0 \leqslant k < l$ and $(m, 2s + 1, s, 1, k)$ satisfies $2s + 1 \leqslant m - k \leqslant \nu + s + 1$. Then*

$$\mathcal{L}_R(m, 2s + 1, s, 1, k; 2\nu + 1 + l, 2\nu + 1) \supset \mathcal{L}_R(m - 1, 2s + \tau_1, s, 0, k; 2\nu + 1 + l, 2\nu + 1),$$

where $\tau_1 = 0, 1$.

Proof. (i) If $\tau_1 = 0$, by the condition we know $\mathcal{M}(m - 1, 2s, s, 0, k; 2\nu + 1 + l, 2\nu + 1) \neq \phi$.

Let $P \in \mathcal{M}(m - 1, 2s, s, 0, k; 2\nu + 1 + l, 2\nu + 1)$, and assume that

$$P = \begin{array}{cc} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{l} (m-1)-k \\ k \end{array} \\ \begin{array}{cc} 2\nu+1 & l \end{array} & \end{array},$$

where $P_{11} S_1 P_{11}^t = \mathcal{M}(m - 1 - k, 2s, s)$, rank $P_{22} = k$, and since $\varepsilon = 0$ we can assume that

$$P_{11} = \begin{array}{cc} \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \end{pmatrix} & \begin{array}{l} 2s \\ 1 \\ m-2-k-2s \end{array} \\ \begin{array}{cc} 2\nu & 1 \end{array} & \end{array}.$$

Let

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ (0^{(2\nu)}, 1) & a \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array}, \quad P_2 = \begin{pmatrix} P_{11} & P_{12} \\ (0^{(2\nu)}, 1) & b \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array}.$$

where $a \neq b$, then $P_i \in \mathcal{M}(m, 2s + 1, s, 1, k; 2\nu + 1 + l, 2\nu + 1)$ $(i = 1, 2)$ and $P = P_1 \cap P_2 \in \mathcal{L}_R(m, 2s + 1, s, 1, k; 2\nu + 1 + l, 2\nu + 1)$.

(ii) If $\tau_1 = 1$, when $m-k < 2s+2$, $\mathcal{M}(m-1, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1) = \phi$, then the conclusion holds obviously.

When $m-k \geqslant 2s+2$, then $\mathcal{M}(m-1, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1) \neq \phi$.

Let $P \in \mathcal{M}(m-1, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1)$, and assume that

$$P = \begin{array}{cc} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{l} (m-1)-k \\ k \end{array} \\ \begin{array}{cc} 2\nu+1 & l \end{array} & \end{array},$$

where

$$P_{11} = \begin{array}{cc} \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \end{pmatrix} & \begin{array}{l} 2s \\ 1 \\ m-2-k-2s \end{array} \\ \begin{array}{cc} 2\nu & 1 \end{array} & \end{array},$$

$\operatorname{rank} P_{22} = k$, and $P_{11}KP_{11}^t = M(m-1-k, 2s+1, s)$. Let

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ (0^{(2\nu)}, 1) & a \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array}, \quad P_2 = \begin{pmatrix} P_{11} & P_{12} \\ (0^{(2\nu)}, 1) & b \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array},$$

where $a \neq b$, then $P_i \in \mathcal{M}(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$ $(i = 1, 2)$ and $P = P_1 \cap P_2 \in \mathcal{L}_R(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$. $\square$

Lemma 3.11. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l$ *and* $(m, 2s+1, s, 1, k)$ *satisfies* $2s+1 \leqslant m-k \leqslant \nu+s+1$. *Then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m-1, 2(s-1)+2, s-1, 0, k; 2\nu+1+l, 2\nu+1). \end{aligned}$$

Proof. By the condition we know $\mathcal{M}(m-1, 2(s-1)+2, s-1, 0, k; 2\nu+1+l, 2\nu+1) \neq \phi$.

Let $P \in \mathcal{M}(m-1, 2(s-1)+2, s-1, 0, k; 2\nu+1+l, 2\nu+1)$, and assume that

$$P = \begin{array}{cc} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{l} (m-1)-k \\ k \end{array} \\ \begin{array}{cc} 2\nu+1 & l \end{array} & \end{array},$$

where

$$P_{11} = \begin{array}{cc} \begin{pmatrix} Q_1 & 0 \\ v & 1 \\ Q_2 & 0 \end{pmatrix} & \begin{array}{l} 2(s-1)+1 \\ 1 \\ m-k-2s-1 \end{array} \\ \begin{array}{cc} 2\nu & 1 \end{array} & \end{array},$$

$\operatorname{rank} P_{22} = k$, $P_{11}KP_{11}^t = \mathcal{M}(m-1-k, 2(s-1)+2, s-1)$. Let

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ (0^{(2\nu)}, 1) & a \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array}, \quad P_2 = \begin{pmatrix} P_{11} & P_{12} \\ (0^{(2\nu)}, 1) & b \\ 0 & P_{22} \end{pmatrix} \begin{array}{l} m-1-k \\ 1 \\ k \end{array},$$

where $a \neq b$, then $P_iS_{1,l}P_i^t$ is cogredient to

$$\begin{pmatrix} 0 & I^{(s-1)} & & & \\ I^{(s-1)} & 0 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

are cogredient, then $P_iS_{1,l}P_i^t = M(m, 2s+1, s), P_1, P_2 \in \mathcal{M}(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$ and $P = P_1 \cap P_2 \in \mathcal{L}_R(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$. $\square$

Theorem 3.1. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, (\tau, \varepsilon), (\tau_1, \varepsilon_1) = (0,0), (1,0), (1,1)$ *and* $(2,0)$, *assume that* $(m, 2s+\tau, s, \varepsilon, k)$ *satisfies*

$$2s + \max\{\tau, \varepsilon\} \leqslant m - k \leqslant \nu + s + [\tau/2] + \varepsilon. \tag{6}$$

(i) *If* $\varepsilon = 0, \varepsilon_1 = 1$, *then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1) \\ &\quad \cap \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, \varepsilon_1, k_1; 2\nu+1+l, 2\nu+1) = \mathbb{F}_q^{(2\nu+1+l)}. \end{aligned}$$

(ii) *If* $\varepsilon = 1, \varepsilon_1 = 0$, *then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, \varepsilon_1, k_1; 2\nu+1+l, 2\nu+1) \end{aligned}$$

iff

$$\left.\begin{array}{l} k_1 \leqslant k < l, \\ (m-k) - (m_1 - k_1) \geqslant s - s_1 + 1 \geqslant 1, \quad \textit{if } \tau_1 = 0, 1, \\ (m-k) - (m_1 - k_1) \geqslant s - s_1 \geqslant 1, \quad \textit{if } \tau_1 = 2. \end{array}\right\} \tag{7}$$

(iii) *If* $\varepsilon = \varepsilon_1 = 1$, *then* $(m_1, 2s_1+1, s_1, 1, k_1)$ *satisfy* $2s_1 + 1 \leqslant m_1 - k_1 \leqslant \nu + s_1 + 1$, *then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m_1, 2s_1+1, s_1, 1, k_1; 2\nu+1+l, 2\nu+1) \end{aligned}$$

iff

$$k_1 \leqslant k < l, (m-k) - (m_1 - k_1) \geqslant s - s_1 \geqslant 0. \tag{8}$$

(iv) *If* $\varepsilon = \varepsilon_1 = 0, \tau = 0$ *and* $\tau_1 = 1, 2$, *then*

$$\mathcal{L}_R(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1) \not\supseteq \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)$$

expect $\mathcal{M}(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1) = \phi$.

(v) *If* $\varepsilon = \varepsilon_1 = 0, \tau \neq 0$ *or* $\tau = \tau_1 = 0, \mathcal{M}(m_1, 2s_1+\tau_1, s_1, 0, k_1) \neq \phi$, *then*

$$\begin{aligned} &\mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1) \\ &\quad \supset \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1) \end{aligned}$$

iff

$$k_1 \leqslant k < l, \quad (m-k) - (m_1 - k_1) \geqslant s - s_1 + \lceil (\tau - \tau_1)/2 \rceil \geqslant \lceil (\tau - \tau_1)/2 \rceil, \tag{9}$$

where $\lceil x \rceil$ *is the least integer not less to* x.

Proof. Firstly, (i) and (iv) are obvious,and the proof way of (ii), (iii) and (v) is similar. So we only prove (v) for example.

$\Leftarrow$ We differ the three case as follows:

(a) $\tau - \tau_1 = 0$. The formula (9) is changed into (8).
Let $s - s_1 = t, k - k_1 = h, m - m_1 = t + h + t'$, $(t, h, t' \geqslant 0)$, by Lemma 3.8

$$\begin{aligned}
&\mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m-1, 2s+\tau, s, 0, k-1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m-h, 2s+\tau, s, 0, k-h; 2\nu+1+l, 2\nu+1)\\
&= \mathcal{L}_R(m_1+t+t', 2s+\tau, s, 0, k_1; 2\nu+1+l, 2\nu+1),
\end{aligned}$$

by Lemma 3.4

$$\begin{aligned}
&\mathcal{L}_R(m_1+t+t', 2s+\tau, s, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m_1+t+t'-1, 2(s-1)+\tau, s-1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m_1+t', 2(s-t)+\tau, s-t, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&= \mathcal{L}_R(m_1+t', 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1),
\end{aligned}$$

by Lemma 3.3

$$\begin{aligned}
&\mathcal{L}_R(m_1+t', 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m_1+t'-1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m_1+t'-t', 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&= \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1).
\end{aligned}$$

(b) $\tau - \tau_1 \geqslant 1$. The formula (9) is changed into

$$k_1 \leqslant k < l, \quad (m-k) - (m_1-k_1) \geqslant s - s_1 + 1 \geqslant 1.$$

Let $s - s_1 = t, k - k_1 = h, m - m_1 = t + h + t'$, $(t, h \geqslant 0, t' \geqslant 1)$, by Lemmas 3.8 and 3.4

$$\begin{aligned}
&\mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m-1, 2s+\tau, s, 0, k-1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m-h, 2s+\tau, s, 0, k-h; 2\nu+1+l, 2\nu+1)\\
&= \mathcal{L}_R(m_1+t+t', 2s+\tau, s, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m_1+t+t'-1, 2(s-1)+\tau, s-1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m_1+t', 2(s-t)+\tau, s-t, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&= \mathcal{L}_R(m_1+t', 2s_1+\tau, s_1, 0, k_1; 2\nu+1+l, 2\nu+1),
\end{aligned}$$

when $\tau - \tau_1 = 1$, by Lemma 3.5

$$\begin{aligned}
&\mathcal{L}_R(m_1+t', 2s_1+\tau, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m_1+t'-1, 2s_1+(\tau-1), s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1),
\end{aligned}$$

when $\tau - \tau_1 = 2$, by Lemmas 3.6 and 3.3

$$\begin{aligned}
&\mathcal{L}_R(m_1+t', 2s_1+2, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \mathcal{L}_R(m_1+t'-1, 2s_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1)\\
&\supset \cdots \supset \mathcal{L}_R(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1).
\end{aligned}$$

(c) $\tau - \tau_1 = -1$, i.e. $\tau = 1, \tau_1 = 2$. The formula (9) is changed into

$$k_1 \leqslant k < l, \quad (m-k) - (m_1-k_1) \geqslant s - s_1 \geqslant 1.$$

Let $s - s_1 = t, k - k_1 = h, m - m_1 = t + h + t', (t, h \geqslant 0, t' \geqslant 1)$, by Lemmas 3.8 and 3.4

$$\begin{aligned}&\mathcal{L}_R(m, 2s + 1, s, 0, k; 2\nu + 1 + l, 2\nu + 1)\\ &\ \supset \mathcal{L}_R(m_1 + t + t', 2s + 1, s, 0, k_1; 2\nu + 1 + l, 2\nu + 1)\\ &\ \supset \mathcal{L}_R(m_1 + 1 + t', 2(s_1 + 1) + 1, s_1 + 1, 0, k_1; 2\nu + 1 + l, 2\nu + 1).\end{aligned}$$

Since $(m_1 + 1 + t', 2(s_1 + 1) + 1, s_1 + 1, 0, k_1)$ satisfies condition (5), by Lemmas 3.7 and 3.3

$$\begin{aligned}&\mathcal{L}_R(m_1 + 1 + t', 2(s_1 + 1) + 1, s_1 + 1, 0, k_1; 2\nu + 1 + l, 2\nu + 1)\\ &\ \supset \mathcal{L}_R(m_1 + t', 2s_1 + 2, s_1, 0, k_1; 2\nu + 1 + l, 2\nu + 1)\\ &\ \supset \mathcal{L}_R(m_1, 2s_1 + 2, s_1, 0, k_1; 2\nu + 1 + l, 2\nu + 1).\end{aligned}$$

$\Rightarrow$ Suppose that

$$\begin{aligned}&\mathcal{L}_R(m, 2s + \tau, s, 0, k; 2\nu + 1 + l, 2\nu + 1)\\ &\ \supset \mathcal{L}_R(m_1, 2s_1 + \tau_1, s_1, 0, k_1; 2\nu + 1 + l, 2\nu + 1).\end{aligned}$$

By the condition we know $\mathcal{M}(m_1, 2s_1 + \tau_1, s_1, 0, k_1; 2\nu + 1 + l, 2\nu + 1) \neq \phi$. So

$$\begin{aligned}&\mathcal{L}_R(m, 2s + \tau, s, 0, k; 2\nu + 1 + l, 2\nu + 1)\\ &\ \supset \mathcal{M}(m_1, 2s_1 + \tau_1, s_1, 0, k_1; 2\nu + 1 + l, 2\nu + 1).\end{aligned}$$

For $Q \in \mathcal{M}(m_1, 2s_1 + \tau_1, s_1, 0, k_1; 2\nu + 1 + l, 2\nu + 1)$ and $Q \neq \mathbb{F}_q^{(2\nu+1+l)}$, since Q is the intersection of subspaces of $\mathcal{M}(m, 2s + \tau, s, 0, k; 2\nu + 1 + l, 2\nu + 1)$, we distinguish the following two cases:

(a) $\tau = \tau_1 = 0$. The formula (9) is changed into formula (8), by Lemma 3.2, and the necessity of Theorem 1 in Ref. [10] we know the formula (8) hold.

(b) $\tau \neq 0$. There is $P \in \mathcal{M}(m, 2s + \tau, s, 0, k; 2\nu + 1 + l, 2\nu + 1)$ such that $Q \subset P$, so we have $k_1 = \dim(Q \cap E) \leqslant \dim(P \cap E) = k$, by Theorem 4.24 in Ref. [12], we know there is an $\varepsilon = 0, 1$ such that

$$(\tau_1, \varepsilon_1) = \begin{cases}(0,0), (1,0), (1,1), (2,0) & \text{if } \tau = 1\\ (0,0), (0,1), (1,0), (2,0), (2,1) & \text{if } \tau = 2\end{cases}, \tag{10}$$

and

$$\begin{aligned}&\max\{0, m_1 - k_1 - s - s_1 - [(\tau_1 + \tau - 1)/2] - \varepsilon_1\}\\ &\ \leqslant \min\{m - k - 2s - \tau, m_1 - k_1 - 2s_1 - \max\{\tau_1, \varepsilon_1\}\}.\end{aligned} \tag{11}$$

Considering the condition $\varepsilon = \varepsilon_1 = 0$, in the following we only study the case $\varepsilon_1 = 0$, where ε_1 is in formulas (10) and (11).

(b-1) $\tau = 1$. We differ the three cases $\tau_1 = 0, 1, 2$ as follows:

(b-1.1) $\tau_1 = 0$. The formula (11) is changed into

$$\max\{0, m_1 - k_1 - s - s_1\} \leqslant \min\{m - k - 2s - 1, m_1 - k_1 - 2s_1\}.$$

By $m_1 - k_1 - s - s_1 \leqslant m - k - 2s - 1$, then $(m - k) - (m_1 - k_1) \geqslant s - s_1 + 1$, and by $m_1 - k_1 - s - s_1 \leqslant m_1 - k_1 - 2s_1$,then $s - s_1 \geqslant 0$, thus

$$(m - k) - (m_1 - k_1) \geqslant s - s_1 + 1 \geqslant 1.$$

(b-1.2) $\tau_1 = 1$. The formula (11) is changed into

$$\max\{0, m_1 - k_1 - s - s_1\} \leqslant \min\{m - k - 2s - 1, m_1 - k_1 - 2s_1 - 1\}.$$

From $m_1 - k_1 - s - s_1 \leqslant m - k - 2s - 1$, then $(m - k) - (m_1 - k_1) \geqslant s - s_1 + 1$, and from $m_1 - k_1 - s - s_1 \leqslant m_1 - k_1 - 2s_1 - 1$, then $s - s_1 \geqslant 1$, thus

$$(m-k)-(m_1-k_1) \geqslant s-s_1+1 \geqslant 2.$$

(b-1.3) $\tau_1 = 2$. The formula (11) is changed into

$$\max\{0, m_1-k_1-s-s_1-1\} \leqslant \min\{m-k-2s-1, m_1-k_1-2s_1-2\}.$$

By $m_1-k_1-s-s_1-1 \leqslant m-k-2s-1$, then $(m-k)-(m_1-k_1) \geqslant s-s_1$, and by $m_1-k_1-s-s_1-1 \leqslant m_1-k_1-2s_1-2$, then $s-s_1 \geqslant 1$, thus

$$(m-k)-(m_1-k_1) \geqslant s-s_1 \geqslant 1.$$

So the formula (11) holds, when $\tau = 1$.

(b-2) $\tau = 2$. We differ the three cases $\tau_1 = 0, 1, 2$ as follows:

(b-2.1) $\tau_1 = 0$. The formula (11) is changed into

$$\max\{0, m_1-k_1-s-s_1\} \leqslant \min\{m-k-2s-2, m_1-k_1-2s_1\}.$$

Thus

$$(m-k)-(m_1-k_1) \geqslant s-s_1+1 \geqslant 1.$$

(b-2.2) $\tau_1 = 1$. The formula (11) is changed into

$$\max\{0, m_1-k_1-s-s_1-1\} \leqslant \min\{m-k-2s-2, m_1-k_1-2s_1-1\}.$$

Thus

$$(m-k)-(m_1-k_1) \geqslant s-s_1+1 \geqslant 1.$$

(b-2.3) $\tau_1 = 2$. The formula (11) is changed into

$$\max\{0, m_1-k_1-s-s_1-1\} \leqslant \min\{m-k-2s-2, m_1-k_1-2s_1-2\}.$$

Thus

$$(m-k)-(m_1-k_1) \geqslant s-s_1 \geqslant 0.$$

From the discussion above, Theorem 3.1 is proved completely. □

4. Characterization of subspaces in $\mathbb{F}_q^{(2\nu+1+l)}$ contained in $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$

Theorem 4.1. *Let $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, (\tau,\varepsilon) = (0,0), (1,0), (1,1)$ or $(2,0)$. Assume that $(m, 2s+\tau, s, \varepsilon, k)$ satisfies condition (6).*

(i) *If $(\tau,\varepsilon) = (0,0)$, then $\mathcal{L}_R(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)$ consists of $\mathbb{F}_q^{(2\nu+1+l)}$ and all the subspaces of type $(m_1, 2s_1, s_1, 0, k_1)$, where $(m_1, 2s_1, s_1, 0, k_1)$ satisfies condition (9).*

(ii) *If $(\tau,\varepsilon) = (1,1)$, then $\mathcal{L}_R(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$ consists of $\mathbb{F}_q^{(2\nu+1+l)}$, all the subspaces of type $(m_1, 2s_1+\tau_1, s_1, 0, k_1)$, where $(m_1, 2s_1+\tau_1, s_1, 0, k_1)$ satisfies condition (7), and all the subspaces of type $(m_1, 2s_1+1, s_1, 1, k_1)$, where $(m_1, 2s_1+1, s_1, 1, k_1)$ satisfies condition (8).*

(iii) *If $(\tau,\varepsilon) = (1,0)$, or $(2,0)$, then $\mathcal{L}_R(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$ consists of $\mathbb{F}_q^{(2\nu+1+l)}$ and all the subspaces of type $(m_1, 2s_1+\tau_1, s_1, 0, k_1)$, where $(m_1, 2s_1+\tau_1, s_1, 0, k_1)$ satisfies condition (9).*

Proof. We only prove (i) for example, since (ii) and (iii) can be obtained in the similar way.

By the agreement, $\mathbb{F}_q^{(2\nu+1+l)} \in \mathcal{L}_R(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1)$.

Let Q be a subspace of type $(m_1, 2s_1, s_1, 0, k_1)$, where $(m_1, 2s_1, s_1, 0, k_1)$ satisfies condition (8), by Theorem 3.1 we have

$$Q \in \mathcal{L}_R(m_1, 2s_1, s_1, 0, k_1;\ 2\nu+1+l, 2\nu+1) \subset \mathcal{L}_R(m, 2s, s, 0, k;\ 2\nu+1+l, 2\nu+1).$$

Conversely, if $Q \in \mathcal{L}_R(m, 2s, s, 0, k;\ 2\nu+1+l, 2\nu+1)$, $Q \neq \mathbb{F}_q^{(2\nu+1+l)}$ and Q is the subspace of type $(m_1, 2s_1, s_1, 0, k_1)$, since Q is the intersection of subspaces in $\mathcal{M}(m, 2s, s, 0, k;\ 2\nu+1+l, 2\nu+1)$, there is $P \in \mathcal{M}(m, 2s, s, 0, k;\ 2\nu+1+l, 2\nu+1)$ such that $Q \subset P$, by the proof of necessity of Theorem 1, we know $(m_1, 2s_1, s_1, 0, k_1)$ satisfies condition (8). $\square$

Corollary 4.1. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, (\tau, \varepsilon) = (0,0), (1,0), (1,1)$ *or* $(2,0)$. *Assume that* $(m, 2s+\tau, s, \varepsilon, k)$ *satisfies condition* (6), *then*

$$\{0\} \in \mathcal{L}_R(m, 2s+\tau, s, 0, k;\ 2\nu+1+l, 2\nu+1)$$

and is the maximal element;

$$\{0\} \in \mathcal{L}_R(m, 2s+1, s, 1, k;\ 2\nu+1+l, 2\nu+1)$$

and is the maximal element.

Corollary 4.2. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l, (\tau, \varepsilon) = (0,0), (1,0), (1,1)$ *and* $(2,0)$. *Assume that* $(m, 2s+\tau, s, \varepsilon, k)$ *satisfies condition* (6). *If* $P \in \mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+1+l, 2\nu+1), P \neq \mathbb{F}_q^{(2\nu+1+l)}$, *and* $Q \subset P$, *then* $Q \in \mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+1+l, 2\nu+1)$.

5. A theorem of relation of inclusion between different subspaces in $\mathbb{F}_q^{(2\nu+1+l)}$

Theorem 5.1. *Let V and U be the subspaces of type* $(m_1, 2s_1+\tau_1, s_1, \varepsilon_1, k_1)$ *and type* $(m_2, 2s_2+\tau_2, s_2, \varepsilon_2, k_2)$ *in* $\mathbb{F}_q^{(2\nu+1+l)}$ *respectively, and* $U \subset V$, *then* $(\tau_2, \varepsilon_2;\ \tau_1, \varepsilon_1)^t$ *takes values in the table as follows.*

No.	1	2	3	4	5	6	7	8	9	10	11
τ_2	0	0	0	0	1	1	1	1	2	2	2
ε_2	0	0	0	0	0	0	0	1	0	0	0
τ_1	0	1	1	2	1	1	2	1	1	1	2
ε_1	0	0	1	0	0	1	0	1	0	1	0

Proof. It can be proved by Theorem 4.24 in Ref. [12]. $\square$

Theorem 5.2. *Let V and U be the subspaces of type* $(m_1, 2s_1+\tau_1, s_1, \varepsilon_1, k_1)$ *and type* $(m_2, 2s_2+\tau_2, s_2, \varepsilon_2, k_2)$ *in* $\mathbb{F}_q^{(2\nu+1+l)}$ *respectively*, $(m_1, 2s_1+\tau_1, s_1, \varepsilon_1, k_1)$ *and* $(m_2, 2s_2+\tau_2, s_2, \varepsilon_2, k_2)$ *satisfy condition* (3) *and* (4). *If* $U \subset V$, *then there exists the matrix representation of subspace V (we still denote by V) such that*

(i) *When* $(\tau_2, \varepsilon_2;\ \tau_1, \varepsilon_1)^t$ *takes values from the* 1*st,* 2*nd,* 3*rd,* 6*th,* 7*th,* 8*th,* 9*th and* 10*th column of the table in Theorem* 5.1, *we have*

$$VS_{1,l}V^t = \left[K_{2s_2}, \Lambda_2, K_{(2s_3,\sigma_1)}, K_{2s_4}, \Lambda_4, 0^{(\sigma)}, 0^{(k_2)}, 0^{(k_1-k_2)}\right], \tag{12}$$

where $s_3, s_4 \geqslant 0, \sigma_1 = m_2 - k_2 - 2s_2 - s_3 - \tau_2 \geqslant 0, \sigma = (m_1 - k_1) - (m_2 - k_2) - s_3 - 2s_4 - \tau_4 \geqslant 0$,

$$\Lambda_2 = \begin{cases} \phi, & \text{if } \tau_2 = 0, \\ (1), & \text{if } \tau_2 = 1, \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \tau_2 = 2, \end{cases}$$

$$\Lambda_4 = \begin{cases} \phi, & \text{if } \tau_4 = 0, \\ (1), & \text{if } \tau_4 = 1, \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \tau_4 = 2, \end{cases}$$

$$\textit{where}\quad \tau_4=\begin{cases}0, & \textit{if } \tau_1=\tau_2,\\ 1, & \textit{if } |\tau_1-\tau_2|=1,\\ 2, & \textit{if } |\tau_1-\tau_2|=2,\end{cases}$$

$$K_{(2s_3,\sigma_1)}=\begin{pmatrix}0 & 0 & I^{(s_3)}\\ 0 & 0^{(\sigma_1)} & 0\\ I^{(s_3)} & 0 & 0\end{pmatrix},$$

besides

$$(m_1-k_1)-(m_2-k_2)\geqslant \begin{cases} s-s_1+\lceil(\tau-\tau_1)/2\rceil\geqslant\lceil|\tau-\tau_1|/2\rceil,\\ \quad (\tau_1,\varepsilon_1)=(1,1),(\tau_2,\varepsilon_2)=(0,0)\ \textit{or}\ (2,0),\\ s-s_1+\lceil(\tau-\tau_1)/2\rceil+|\varepsilon_1-\varepsilon_2|\geqslant\lceil|\tau-\tau_1|/2\rceil+|\varepsilon_1-\varepsilon_2|,\\ \textit{others}\end{cases} \tag{13}$$

(ii) *When* $(\tau_2,\varepsilon_2;\,\tau_1,\varepsilon_1)^t$ *takes values from the* 4*th column of the table in Theorem* 5.1, *we have*

$$VS_{1,l}V^t=\left[K_{2s_2},K_{(2s_3,\sigma_1)},K_{2s_4},\begin{pmatrix}0&1\\1&1\end{pmatrix},0^{(\sigma-2)},0^{(k_2)},0^{(k_1-k_2)}\right], \tag{14}$$

or

$$VS_{1,l}V^t=\left[K_{2s_2},\begin{pmatrix}0 & 0 & I^{(s_3)}\\ 0 & 0^{(\sigma_1)} & 0\\ I^{(s_3)} & 0 & E_3\end{pmatrix},K_{2s_4},0^{(\sigma)},0^{(k_2)},0^{(k_1-k_2)}\right], \tag{15}$$

where $s_4\geqslant 0,\sigma_1=m_2-k_2-2s_2-s_3\geqslant 0,E_3=\left[0^{(s_3-1)},1\right],k_1\geqslant k_2$, *and*

$$\sigma=\begin{cases}(m_1-k_1)-(m_2-k_2)-s_3-2s_4\geqslant 2,s_3\geqslant 0 & \textit{in formula } (14),\\ (m_1-k_1)-(m_2-k_2)-s_3-2s_4\geqslant 0,s_3\geqslant 1 & \textit{in formula } (15),\end{cases}$$

besides $(m_1-k_1)-(m_2-k_2)\geqslant s_1-s_2+2\geqslant 2$ *or* $(m_1-k_1)-(m_2-k_2)\geqslant s_1-s_2+1\geqslant 1$.

(iii) *When* $(\tau_2,\varepsilon_2;\,\tau_1,\varepsilon_1)^t$ *takes values from the* 5*th and* 11*th column of the table in Theorem* 5.1, *we have*

$$VS_{1,l}V^t=[K_{2s_2},\Lambda_2,K_{(2s_3,\sigma_1)},K_{2s_4},0^{(\sigma)},0^{(k_2)},0^{(k_1-k_2)}], \tag{16}$$

or

$$VS_{1,l}V^t=\left[K_{2s_2},\Lambda_2,K_{(2s_3,\sigma_1)},K_{2s_4},\begin{pmatrix}0&1\\1&1\end{pmatrix},0^{(\sigma-2)},0^{(k_2)},0^{(k_1-k_2)}\right], \tag{17}$$

where

$$\Lambda_2=\begin{cases}\phi, & \textit{if } \tau_1=\tau_2=1\\ \begin{pmatrix}0&1\\1&1\end{pmatrix}, & \textit{if } \tau_1=\tau_2=2\end{cases},s_3,s_4\geqslant 0,\ \sigma_1=m_2-k_2-2s_2-s_3-\tau_2\geqslant 0,\ k_1\geqslant k_2$$

and

$$\sigma=(m_1-k_1)-(m_2-k_2)-s_3-2s_4\geqslant\begin{cases}0, & \textit{in formula } (16),\\ 2, & \textit{in formula } (17),\end{cases}$$

besides $(m_1-k_1)-(m_2-k_2)\geqslant s_1-s_2\geqslant 0$ *or* $(m_1-k_1)-(m_2-k_2)\geqslant s_1-s_2+1\geqslant 2$.

In all the cases (i)–(iii), *we have* $U=\langle v_1,v_2,\ldots,v_{m_2-k_2},v_{m_1-k_1+1},v_{m_1-k_1+2},\ldots,v_{m_1-k_1+k_2}\rangle$, $V=\langle v_1,v_2,\ldots,v_{m_2-k_2},\ldots,v_{m_1-k_1},\ldots,v_{m_1}\rangle$, $\langle v_{m_1-k_1+1},v_{m_1-k_1+2},\ldots,v_{m_1}\rangle\subset E$, v_i *is the ith row vector of* V.

Proof. We only show the proof of $(\tau_2,\varepsilon_2;\,\tau_1,\varepsilon_1)^t$ takes values from the 5th column of the table, others can be obtained in the similar way.

Let

$$U = \begin{array}{cc} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} & \begin{array}{l} m_2-k_2 \\ k_2 \end{array} \\ \begin{array}{cc} 2\nu+1 & l \end{array} & \end{array},$$

where $U_{11}S_1U_{11}^t = M(m_2-k_2, 2s_2+1, s_2)$, $\operatorname{rank} U_{11} = m_2-k_2$, $\operatorname{rank} U_{22} = k_2$. Since $U \subset V$, $U \cap E \subset V \cap E$, $k_1 = \dim(V \cap E) \geqslant \dim(U \cap E) = k_2$, so there exists the matrix representation of V

$$V = \begin{array}{cc} \begin{pmatrix} U_{11} & U_{12} \\ U'_{11} & U'_{12} \\ 0 & U_{22} \\ 0 & U'_{22} \end{pmatrix} & \begin{array}{l} m_2-k_2 \\ (m_1-k_1)-(m_2-k_2) \\ k_2 \\ k_1-k_2 \end{array} \\ \begin{array}{cc} 2\nu+1 & l \end{array} & \end{array},$$

where U_{11} represents the subspace of type $(m_2-k_2, 2s_2+1, s_2, 0)$ in $\mathbb{F}_q^{(2\nu+1)}$, $\begin{pmatrix} U_{11} \\ U'_{11} \end{pmatrix}$ represents the subspace of type $(m_1-k_1, 2s_1+1, s_1, 0)$ in $\mathbb{F}_q^{(2\nu+1)}$ and $U_{11} \subset \begin{pmatrix} U_{11} \\ U'_{11} \end{pmatrix}$. By Theorem 7.28 in Ref. [9], there exists $(m_1-k_1) \times (m_1-k_1)$ nonsingular matrix R_1 such that

$$R_1 \begin{pmatrix} U_{11} \\ U'_{11} \end{pmatrix} = \begin{pmatrix} \widetilde{U}_{11} \\ \widetilde{U}'_{11} \end{pmatrix},$$

where $\widetilde{U}_{11}$ still is the matrix representation of the subspace U_{11}, and

$$\begin{pmatrix} \widetilde{U}_{11} \\ \widetilde{U}'_{11} \end{pmatrix} S_1 \begin{pmatrix} \widetilde{U}_{11} \\ \widetilde{U}'_{11} \end{pmatrix}^t = [K_{2s_2}, 1, K_{(2s_3,\sigma_1)}, K_{2s_4}, 0^{(\sigma)}],$$

where $s_3, s_4 \geqslant 0, \sigma_1 = m_2-k_2-2s_2-s_3-1 \geqslant 0, \sigma = (m_1-k_1)-(m_2-k_2)-s_3-2s_4 \geqslant 0$ and $s_1 = s_2+s_3+s_4, (m_1-k_1)-(m_2-k_2) \geqslant s_1-s_2 \geqslant 0$. Let

$$R = \begin{pmatrix} R_1 & & \\ & I & \\ & & I \end{pmatrix},$$

then

$$V = R\begin{pmatrix} U_{11} & U_{12} \\ U'_{11} & U'_{12} \\ 0 & U_{22} \\ 0 & U'_{22} \end{pmatrix} = \begin{pmatrix} \widetilde{U}_{11} & \widetilde{U}_{12} \\ \widetilde{U}'_{11} & \widetilde{U}'_{12} \\ 0 & U_{22} \\ 0 & U'_{22} \end{pmatrix}$$

still is the matrix representation of the subspace V, $\begin{pmatrix} \widetilde{U}_{11} & \widetilde{U}_{12} \\ 0 & U_{22} \end{pmatrix}$ still is the matrix representation of the subspace U and

$$VS_{1,l}V^t = [K_{2s_2}, 1, K_{(2s_3,\sigma_1)}, K_{2s_4}, 0^{(\sigma)}, 0^{(k_2)}, 0^{(k_1-k_2)}],$$

i.e. formula (16) holds, besides $U = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_1-k_1+1}, v_{m_1-k_1+2}, \ldots, v_{m_2-k_1+k_2}\rangle$, v_i is the ith row vector of V.

Note: the conclusion about $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+1+l, 2\nu+1)$ still holds for $\mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+1+l, 2\nu+1)$, if the maximal element is changed into minimal element. □

6. The rank function of lattice $\mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+1+l, 2\nu+1)$ and $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+1+l, 2\nu+1)$

Theorem 6.1. *Let* $2\nu+1+l > m \geqslant 1, 0 \leqslant k < l$ *and* $(m, 2s+\tau, s, \varepsilon, k)$ *satisfies*

$$\begin{cases} (\tau,\varepsilon) = (0,0), (1,0), (1,1), (2,0), \\ 2s+\max\{\tau,\varepsilon\} \leqslant m-k \leqslant \nu+s+[\tau/2]+\varepsilon. \end{cases} \tag{18}$$

For any $X \in \mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$, *define*

$$r(X) = \begin{cases} \dim X & \text{if } X \neq \mathbb{F}_q^{(2\nu+1+l)}, \\ m+1 & \text{if } X = \mathbb{F}_q^{(2\nu+1+l)}, \end{cases} \tag{19}$$

then $r : \mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1) \to \mathbb{N}$ *is the rank function of lattice* $\mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$.

Proof. Clearly, $\{0\} \in \mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ and is the minimal element, condition (i) of Proposition 2.1 holds for the function r. Now assume that $U, V \in \mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ and $U \leqslant V$. Suppose that $r(V) - r(U) > 1$, we will show that $U < \cdot V$ does not hold, i.e. condition (ii) of Proposition 2.1 holds for the function r.

First, if $V = \mathbb{F}_q^{(2\nu+1+l)}$, then $r(V) = m+1$, $\dim U < m$, and U is the intersection of subspaces in $\mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$, so there is a subspace $W \in \mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ such that $U < W < V$, i.e $U < \cdot V$ does not hold.

Assumed that $V \neq \mathbb{F}_q^{(2\nu+1+l)}$,we distinguish the following two cases:

(a) $\varepsilon = 0$. Let V and U be of type $(m_1, 2s_1+\tau_1, s_1, 0, k_1)$ and $(m_2, 2s_2+\tau_2, s_2, 0, k_2)$ respectively, and $m_1 - m_2 \geqslant 2$, by Theorem 5.1, $(\tau_1, \tau_2) = (0,0), (1,0), (2,0), (1,1), (2,1), (1,2)$ or $(2,2)$. When $(\tau_1, \tau_2) = (0,0), (1,0), (2,1), (1,2)$, the formula (12) holds; when $(\tau_1, \tau_2) = (2,0)$, the formula (14) or (15) hold; when $(\tau_1, \tau_2) = (1,1), (2,2)$, the formula (16) or (17) hold,and in all the cases we have $U = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_1-k_1+1}, v_{m_1-k_1+2}, \ldots, v_{m_1-k_1+k_2}\rangle$, $V = \langle v_1, v_2, \ldots, v_{m_1}\rangle$, v_i is the ith row vector of V.

Firstly, we discuss the cases $(\tau_1, \tau_2) = (0,0), (1,0), (2,1)or(1,2)$.We distinguish the following two cases:

(a.1) $k_2 < k_1$. Let $W = \langle v_1, v_2, \ldots, v_{m_1-1}\rangle$, then W is the subspace of type $(m_1-1, 2s_1+\tau_1, s_1, 0, k_1-1)$, since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

(a.2) $k_2 = k_1$. The case $(m_1-k_1) - (m_2-k_2) - s_3 - 2s_4 - \tau_4 = s_3 = s_4 = 0$ does not exist. We distinguish the following three cases:

(a.2.1) $(m_1-k_1) - (m_2-k_2) - s_3 - 2s_4 - \tau_4 > 0$. Let $W = \langle v_1, v_2, \ldots, v_{m_1-k_1-1}, v_{m_1-k_1+1} \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2s_1+\tau_1, s_1, 0, k_1)$, since $W \subset V$,$V \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

(a.2.2) $s_3 > 0$. Let $W = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_2-k_2+2}, \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2(s_1-1)+\tau_1, s_1-1, 0, k_1)$, since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

(a.2.3) $(m_1-k_1) - (m_2-k_2) - s_3 - 2s_4 - \tau_4 = s_3 = 0$. We have $m_1 - m_2 - \tau_4 = 2s_4$. Since $\tau_4 = 0$ or 1, we have $s_4 > 0$. Let $W = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_2-k_2+2}, \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2(s_1-1)+\tau_1, s_1-1, 0, k_1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

Secondly, we discuss the case $(\tau_1, \tau_2) = (2,0)$.

If $k_2 < k_1$, we can obtain that $U < \cdot V$ does not hold in the similar way of (a.1), so we only show the case $k_2 = k_1$.

When the formula (15) holds, we have $s_3 \geqslant 1$. Let $W = \langle v_1, v_2, \ldots, v_{m_2-k_2+s_3-1}, v_{m_2-k_2+s_3+1}, \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2(s_1-1)+\tau_1, s_1-1, 0, k_1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

When the formula (14) holds, if $(m_1-k_1) - (m_2-k_2) - s_3 - 2s_4 > 2$, or $s_3 > 0$, or $s_4 > 0$, by the similar discussion of (a.2), there is $W \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$ such that $U < W < V$; if $(m_1-k_1) - (m_2-k_2) - s_3 - 2s_4 = 2$, let $W = \langle v_1, v_2, \ldots, v_{m_1-1}\rangle$, then W is the subspace of type $(m_1-1, 2s_1, s_1, 0, k_1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+\tau, s, 0, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

For the case $(\tau_1, \tau_2) = (2,2)$ or $(1,1)$ which can be obtained in the same way, so we omit it here.

(b) $\varepsilon = 1$. We have $(\tau, \varepsilon) = (1,1)$, let V be the subspace of type $(m_1, 2s_1+\tau_1, s_1, 1, k_1)$, by the Theorem 5.1 we know $(\tau_2, \varepsilon_2, \tau_1, \varepsilon_1)$ takes values from the 3th, 6th, 8th, and the 10th column of table, and the formula (12) holds, and in all the cases we have $U = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_1-k_1+1}, v_{m_1-k_1+2}, \ldots, v_{m_1-k_1+k_2}\rangle$, $V = \langle v_1, v_2, \ldots, v_{m_1}\rangle$, v_i is the ith row vector of V. We distinguish the following two cases:

(b.1) $k_2 < k_1$. Let $W = \langle v_1, v_2, \ldots, v_{m_1-1}\rangle$, then W is the subspace of type $(m_1-1, 2s_1+1, s_1, 1, k_1-1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

(b.2) $k_2 = k_1$. When $m_1 - m_2 \geqslant 2$, and $|\tau_1 - \tau_2| \leqslant 1$, the case $(m_1-k_1)-(m_2-k_2)-s_3-2s_4-\tau_4 = s_3 = s_4 = 0$ does not exist. We distinguish the following three cases:

(b.2.1) $(m_1-k_1)-(m_2-k_2)-s_3-2s_4-\tau_4 > 0$. Let $W = \langle v_1, v_2, \ldots, v_{m_1-k_1-1}, v_{m_1-k_1+1}, \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2s_1+1, s_1, 1, k_1-1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

(b.2.2) $s_3 > 0$. Let $W = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_2-k_2+2}, \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2(s_1-1)+1, s_1-1, 1, k_1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

(b.2.3) $(m_1-k_1)-(m_2-k_2)-s_3-2s_4-\tau_4 = s_3 = 0$. We have $m_1 - m_2 - \tau_4 = 2s_4$. Since $\tau_4 = 0$ or 1, so $s_4 > 0$. Let $W = \langle v_1, v_2, \ldots, v_{m_2-k_2}, v_{m_2-k_2+2}, \ldots, v_{m_1}\rangle$, then W is the subspace of type $(m_1-1, 2(s_1-1)+1, s_1-1, 1, k_1)$. Since $W \subset V$, $V \in \mathcal{L}_O(m, 2s+1, s, 1, k; 2\nu+1+l, 2\nu+1)$, $m_1 - m_2 \geqslant 2$, so $U < W < V$.

Therefore, r defined by formula (19) is a rank function of $\mathcal{L}_O(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$. □

Similarly, we have the theorem as follows:

Theorem 6.2. *Let $2\nu+1+l > m \geqslant 1$, $0 \leqslant k < l$ and $(m, 2s+\tau, s, \varepsilon, k)$ satisfies the condition (18). For any $X \in \mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$, define*

$$r'(X) = \begin{cases} m+1-\dim X & \text{if } X \neq \mathbb{F}_q^{(2\nu+1+l)}, \\ 0 & \text{if } X = \mathbb{F}_q^{(2\nu+1+l)}, \end{cases} \tag{20}$$

then $r' : \mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1) \to \mathbb{N}$ is the rank function of lattice $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$.

7. Characteristic polynomial of lattice $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$

Let $2\nu+1+l > m \geqslant 1$, $0 \leqslant k < l$ and $(m, 2s+\tau, s, \varepsilon, k)$ satisfies the condition (18). Let

$$N(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1) = |\mathcal{M}(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)|$$

be the number of subspaces of type $(m, 2s+\tau, s, \varepsilon, k)$ in $\mathbb{F}_q^{(2\nu+1+l)}$ (see [12]), $g_{m_1} = (t-1)(t-q)(t-q^2)\cdots(t-q^{m_1-1})$ be the Gauss polynomial of degree m_1.

Define the characteristic polynomial of lattice $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ to be

$$\chi(\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1), t) = \sum_{P \in \mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)} \mu(0, P) t^{r'(1)-r'(P)},$$

where 0 and 1 are the minimal and the maximal element of $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ respectively, μ is the Möbius function, and r' is the rank function of $\mathcal{L}_R(m, 2s+\tau, s, \varepsilon, k; 2\nu+1+l, 2\nu+1)$ defined as (20).

Theorem 7.1. *Let $2\nu+1+l > m \geqslant 1$. Assume that $(m, 2s+\tau, s, \varepsilon, k)$ satisfies condition (18), then*

(i) $\chi(\mathcal{L}_R(m, 2s, s, 0, k; 2\nu+1+l, 2\nu+1), t) = \chi(\mathcal{L}_R(m, s, k; 2\nu+l, \nu), t)$
$= t^{m+1} - \sum_{k_1=0}^{k} \sum_{s_1=0}^{s} \sum_{m_1=2s_1+k_1}^{m-(s-s_1)-(k-k_1)} N(m_1, s_1, k_1; 2\nu+l, \nu) g_{m_1}(t).$

(ii) $\chi(\mathcal{L}_R(m, 2s+1, s, 0, k; 2\nu+1+l, 2\nu+1), t) = t^{m+1}$
$- \sum_{\tau_1=0,1\text{or } 2} \left(\sum_{k_1=0}^{k} \sum_{s_1=0}^{s+\lceil (1-\tau_1)/2\rceil - \lceil |1-\tau_1|/2\rceil} \sum_{m_1=2s_1+k_1+1}^{m-(s-s_1)-(k-k_1)-\lceil (1-\tau_1)/2\rceil} \right.$
$\left. \cdot N(m_1, 2s_1+\tau_1, s_1, 0, k_1; 2\nu+1+l, 2\nu+1) g_{m_1}(t) \right).$

(iii) $\chi(\mathcal{L}_R(m,2s+2,s,0,k;2\nu+1+l,2\nu+1),t)=t^{m+1}$
$-\sum_{\tau_1=0,1\text{or}2}\left(\sum_{k_1=0}^{k}\sum_{s_1=0}^{s+\lceil(2-\tau_1)/2\rceil-\lceil|2-\tau_1|/2\rceil}\sum_{m_1=2s_1+k_1+2}^{m-(s-s_1)-(k-k_1)-\lceil(2-\tau_1)/2\rceil}\right.$
$\left.\cdot N(m_1,2s_1+\tau_1,s_1,0,k_1;2\nu+1+l,2\nu+1)g_{m_1}(t)\right)$.

(iv) $\chi(\mathcal{L}_R(m,2s+1,s,1,k;2\nu+1+l,2\nu+1),t)=t^{m+1}$
$-\sum_{\tau_1=0,1\text{or }2}\left(\sum_{k_1=0}^{k}\sum_{s_1=0}^{s+\lceil(1-\tau_1)/2\rceil-\lceil|1-\tau_1|/2\rceil}\sum_{m_1=2s_1+k_1}^{m-(s-s_1)-(k-k_1)-\lceil(1-\tau_1)/2\rceil}\right.$
$\left.\cdot N(m_1,2s_1+\tau_1,s_1,0,k_1;2\nu+1+l,2\nu+1)g_{m_1}(t)\right)$
$+\sum_{k_1=0}^{k}\sum_{s_1=0}^{s}\sum_{m_1=2s_1+k_1}^{m-(s-s_1)-(k-k_1)}N(m_1,2s_1+1,s_1,1,k_1;2\nu+1+l,2\nu+1)g_{m_1}(t)$.

Proof. By Ref. [10] and Lemma 3.2, we can obtain (i). We only show the proof of (ii) for example.

Write $V=\mathbb{F}_q^{(2\nu+1+l)}$, $\mathcal{L}=\mathcal{L}_R(m,2s+1,s,0,k;2\nu+1+l,2\nu+1)$, $\mathcal{L}_0=\mathcal{L}_R(2\nu+1+l,V)$ where $\mathcal{L}_R(2\nu+1+l,V)$ is the lattice partially ordered by reverse inclusion on the set of all subspaces in V. For $P\in\mathcal{L}$, define

$$\mathcal{L}^P=\{Q\in\mathcal{L}|Q\subset P\}=\{Q\in\mathcal{L}|Q\geqslant P\},$$
$$\mathcal{L}_0^P=\{Q\in\mathcal{L}_0|Q\subset P\}=\{Q\in\mathcal{L}_0|Q\geqslant P\}.$$

Clearly, $\mathcal{L}^V=\mathcal{L}$. By Corollary 4.2, $\mathcal{L}^P=\mathcal{L}_0^P$ when $P\neq V$.

$$\chi(\mathcal{L}^V,t)=\chi(\mathcal{L},t)=\sum_{P\in\mathcal{L}}\mu(0,P)t^{r'(1)-r'(P)}.$$

From Möbius inversion formula and Corollary 4.1

$$t^{r'(1)-r'(0)}=t^{m+1}=\sum_{P\in\mathcal{L}^V}\chi(\mathcal{L}^P,t)=\sum_{P\in\mathcal{L}}\chi(\mathcal{L}^P,t),$$
$$\chi(\mathcal{L},t)=\chi(\mathcal{L}^V,t)=t^{m+1}-\sum_{P\in\mathcal{L}\backslash\{V\}}\chi(\mathcal{L}^P,t)=t^{m+1}-\sum_{P\in\mathcal{L}\backslash\{V\}}\chi(\mathcal{L}_0^P,t).$$

By Theorem 4.1, $\mathcal{L}\backslash\{V\}=\{$subspaces of type $(m_1,2s_1+\tau_1,s_1,0,k_1)|(m_1,2s_1+\tau_1,s_1,0\ k_1)$ satisfies condition (9)$\}$. Thus

$$\sum_{P\in\mathcal{L}\backslash\{V\}}\chi(\mathcal{L}_0^P,t)=\sum_{\tau_1=0,1\text{or}2}\left(\sum_{k_1=0}^{k}\sum_{s_1=0}^{s+\lceil(1-\tau_1)/2\rceil-\lceil|1-\tau_1|/2\rceil}\sum_{m_1=2s_1+k_1}^{m-(s-s_1)-(k-k_1)-\lceil(1-\tau_1)/2\rceil}\right.$$
$$\left.\cdot N(m_1,2s_1+\tau_1,s_1,0,k_1;2\nu+1+l,2\nu+1)g_{m_1}(t)\right).$$

Hence

$$\chi(\mathcal{L}_R(m,2s+1,s,0,k;2\nu+1+l,2\nu+1),t)$$
$$=t^{m+1}-\sum_{\tau_1=0,1\text{or}2}\left(\sum_{k_1=0}^{k}\sum_{s_1=0}^{s+\lceil(1-\tau_1)/2\rceil-\lceil|1-\tau_1|/2\rceil}\sum_{m_1=2s_1+k_1+1}^{m-(s-s_1)-(k-k_1)-\lceil(1-\tau_1)/2\rceil}\right.$$
$$\left.\cdot N(m_1,2s_1+\tau_1,s_1,0,k_1;2\nu+1+l,2\nu+1)g_{m_1}(t)\right).\quad\square$$

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